Flexural Analysis and Design of Beams

Chapter 3
Introduction

• Fundamental Assumptions
• Simple case of axial loading
• Same assumptions and ideal concept apply

• This chapter includes analysis and design for flexure, dimensioning cross section and reinforcement

• Shear design, bond anchorage, serviceability in chapters 4, 5, 6.
Bending of Homogeneous beam

- Steel, timber
- Internal forces-normal and tangential
- Normal-bending/flexural stress-bending moment
- Tangential-shear stress-shear force
Fundamental assumptions relating to flexure and shear

1. Plane cross section remain plane
2. Bending stress $f$ at any point depends on the strain at that point
3. Shear stress also depends on cross section and stress-strain diagram. Maximum at neutral axis and zero at extreme fibre. Same horizontal and vertical.
4. The intensity of principal stresses

$$ t = \frac{f}{2} \pm \sqrt{\frac{f^2}{4} + \nu^2} $$

where $f =$ intensity of normal fiber stress
$\nu =$ intensity of tangential shearing stress
5. At neutral axis, only horizontal and vertical shear present—pure shear condition
6. When stress are smaller than proportional limit
   a. Neutral axis = cg
   b. $f = \frac{My}{I}$
   c. $v = \frac{VQ}{It}$
   d. Shear distribution parabolic, max at $na$, zero at outer fibre. For rectangular max=$1.5V/bh$
Reinforced Concrete Beam Behaviour
Video

• See video clips
Stresses elastic, section uncracked

- Tensile stress in concrete is smaller than modulus of rupture
  Transformed section can be used
EXAMPLE 3.1 A rectangular beam has the dimensions (see Fig. 3.2b) \( b = 10 \) in., \( h = 25 \) in., and \( d = 23 \) in. and is reinforced with three No. 8 (No. 25) bars so that \( A_s = 2.37 \) in\(^2\). The concrete cylinder strength \( f'_c \) is 4000 psi, and the tensile strength in bending (modulus of rupture) is 475 psi. The yield point of the steel \( f_y \) is 60,000 psi, the stress-strain curves of the materials being those of Fig. 1.16. Determine the stresses caused by a bending moment \( M = 45 \) ft-kips.

**SOLUTION.** With a value \( n = E_s/E_c = 29,000,000/3,600,000 = 8 \), one has to add to the rectangular outline an area \((n - 1)A_s = 7 \times 2.37 = 16.59 \) in\(^2\), disposed as shown on Fig. 3.4, to obtain the uncracked, transformed section. Conventional calculations show that the location of the neutral axis of this section is given by \( ar{y} = 13.2 \) in. from the top of the section, and its moment of inertia about this axis is 14,740 in\(^4\). For \( M = 45 \) ft-kips = 540,000 in-lb, the concrete compression stress at the top fiber is, from Eq. (3.3),

\[
f_c = \frac{M \bar{y}}{I} = \frac{540,000 \times 13.2}{14,740} = 484 \text{ psi}
\]

and, similarly, the concrete tension stress at the bottom fiber, 11.8 in. from the neutral axis, is

\[
f_{ct} = \frac{540,000 \times 11.8}{14,740} = 432 \text{ psi}
\]
Since this value is below the given tensile bending strength of the concrete, 475 psi, no tension cracks will form, and calculation by the uncracked, transformed section is justified. The stress in the steel, from Eqs. (1.6) and (3.2), is

$$f_s = n \frac{My}{I} = 8 \left( \frac{540,000 \times 9.8}{14,740} \right) = 2870 \text{ psi}$$

By comparing $f_c$ and $f_s$ with the concrete cylinder strength and the yield point, respectively, it is seen that at this stage the actual stresses are quite small compared with the available strengths of the two materials.
Stresses Elastic, Section cracked

- Concrete tensile stress exceeds mod of rupture
- Concrete compressive stress is less than $f'_c / 2$
- Steel stress less than yield
- Assume tension crack up to neutral axis
- Transformed section can still be used
neutral axis, the moment of the tension area about the axis is set equal to the moment of the compression area, which gives

\[ b \frac{(kd)^2}{2} - nA_s(d - kd) = 0 \]  

(3.5)

Having obtained \( kd \) by solving this quadratic equation, one can determine the moment of inertia and other properties of the transformed section as in the preceding case. Alternatively, one can proceed from basic principles by accounting directly for the forces that act on the cross section. These are shown in Fig. 3.5b. The concrete stress, with maximum value \( f_c \) at the outer edge, is distributed linearly as shown. The entire steel area \( A_s \) is subject to the stress \( f_s \). Correspondingly, the total compression force \( C \) and the total tension force \( T \) are

\[ C = \frac{f_c}{2} b kd \quad \text{and} \quad T = A_s f_s \]  

(3.6)
The requirement that these two forces be equal numerically has been taken care of by the manner in which the location of the neutral axis has been determined.

Equilibrium requires that the couple constituted by the two forces \( C \) and \( T \) be equal numerically to the external bending moment \( M \). Hence, taking moments about \( C \) gives

\[
M = Tjd = Asf_sjd
\]  \hspace{1cm} (3.7)

where \( jd \) is the internal lever arm between \( C \) and \( T \). From Eq. (3.7), the steel stress is

\[
f_s = \frac{M}{Asjd}
\]  \hspace{1cm} (3.8)

Conversely, taking moments about \( T \) gives

\[
M = Cjd = \frac{f_c}{2} bkdjd = \frac{f_c}{2} kjbd^2
\]  \hspace{1cm} (3.9)
from which the concrete stress is

\[ f_c = \frac{2M}{kjbd^2} \]  

(3.10)

In using Eqs. (3.6) through (3.10), it is convenient to have equations by which \( k \) and \( j \) may be found directly, to establish the neutral axis distance \( kd \) and the internal lever arm \( jd \). First defining the \textit{reinforcement ratio} as

\[ \rho = \frac{A_s}{bd} \]  

(3.11)

then substituting \( A_s = \rho bd \) into Eq. (3.5) and solving for \( k \), one obtains

\[ k = \sqrt{(\rho n)^2 + 2\rho n - \rho n} \]  

(3.12)

From Fig. 3.5b it is seen that \( jd = d - kd/3 \), or

\[ j = 1 - \frac{k}{3} \]  

(3.13)

Values of \( k \) and \( j \) for elastic cracked section analysis, for common reinforcement ratios and modular ratios, are found in Table A.6 of Appendix A.
The beam of Example 3.1 is subject to a bending moment \( M = 90 \) ft-kips (rather than 45 ft-kips as previously). Calculate the relevant properties and stresses.

**Solution.** If the section were to remain uncracked, the tensile stress in the concrete would now be twice its previous value, that is, 864 psi. Since this exceeds by far the modulus of rupture of the given concrete (475 psi), cracks will have formed and the analysis must be adapted consistent with Fig. 3.5. Equation (3.5), with the known quantities \( b, n, \) and \( A_s \) inserted, gives the distance to the neutral axis \( kd = 7.6 \) in., or \( k = 7.6/23 = 0.33 \). From Eq. (3.13), \( j = 1 - 0.33/3 = 0.89 \). With these values the steel stress is obtained from Eq. (3.8) as \( f_s = 22,300 \) psi, and the maximum concrete stress from Eq. (3.10) as \( f_c = 1390 \) psi.

Comparing the results with the pertinent values for the same beam when subject to one-half the moment, as previously calculated, one notices that (1) the neutral plane has migrated upward so that its distance from the top fiber has changed from 13.2 to 7.6 in.; (2) even though the bending moment has only been doubled, the steel stress has increased from 2870 to 22,300 psi, or about 7.8 times, and the concrete compression stress has increased from 484 to 1390 psi, or 2.9 times; (3) the moment of inertia of the cracked transformed section is easily computed to be 5910 in\(^4\), compared with 14,740 in\(^4\) for the uncracked section. This affects the magnitude of the deflection, as discussed in Chapter 6. Thus, it is seen how radical is the influence of the formation of tension cracks on the behavior of reinforced concrete beams.
Flexural Strength

\[ C = \alpha f_c b c \]

\[ z = d - \beta c \]

\[ T = A_s f_s \]

\[ \epsilon_s = \frac{f_s}{E_s} \]
• Yielding of steel $f_s = f_y$
• Crushing of concrete $\varepsilon_u = 0.003-0.004$
• Either can reach first
• Exact shape not necessary
• Necessary – Total compressive force and location
• $\beta c$ - location from comp face

\[ \alpha = \frac{f_{av}}{f'_c} \quad \text{(3.14)} \]

Then

\[ C = \alpha f'_c b c \quad \text{(3.15)} \]
\( \alpha \) equals 0.72 for \( f'_c \leq 4000 \) psi and decreases by 0.04 for every 1000 psi above 4000 up to 8000 psi. For \( f'_c > 8000 \) psi, \( \alpha = 0.56 \).

\( \beta \) equals 0.425 for \( f'_c \leq 4000 \) psi and decreases by 0.025 for every 1000 psi above 4000 up to 8000 psi. For \( f'_c > 8000 \) psi, \( \beta = 0.325 \).

\[ C = T \quad \text{or} \quad \alpha f'_c b c = A_s f_s \quad (3.16) \]

Also, the bending moment, being the couple of the forces \( C \) and \( T \), can be written as either

\[ M = T z = A_s f_s (d - \beta c) \quad (3.17) \]

or

\[ M = C z = \alpha f'_c b c (d - \beta c) \quad (3.18) \]

For failure initiated by yielding of the tension steel, \( f_s = f_y \). Substituting this value in Eq. (3.16), one obtains the distance to the neutral axis

\[ c = \frac{A_s f_y}{\alpha f'_c b} \quad (3.19a) \]
\[ \alpha \] equals 0.72 for \( f'_c \leq 4000 \) psi and decreases by 0.04 for every 1000 psi above 4000 up to 8000 psi. For \( f'_c > 8000 \) psi, \( \alpha = 0.56 \).

\[ \beta \] equals 0.425 for \( f'_c \leq 4000 \) psi and decreases by 0.025 for every 1000 psi above 4000 up to 8000 psi. For \( f'_c > 8000 \) psi, \( \beta = 0.325 \).
For failure initiated by yielding of the tension steel, \( f_s = f_y \). Substituting this value in Eq. (3.16), one obtains the distance to the neutral axis

\[
c = \frac{A_s f_y}{\alpha f_c'}
\]  
(3.19a)

Alternatively, using \( A_s = \rho bd \), the neutral axis distance is

\[
c = \frac{\rho f_y d}{\alpha f_c'}
\]  
(3.19b)

giving the distance to the neutral axis when tension failure occurs. The nominal moment \( M_n \) is then obtained from Eq. (3.17) with the value for \( c \) just determined, and \( f_s = f_y \); that is,

\[
M_n = \rho f_y bd^2 \left( 1 - \frac{\beta f_y \rho}{\alpha f_c'} \right)
\]  
(3.20a)

With the specific, experimentally obtained values for \( \alpha \) and \( \beta \) given previously, this becomes

\[
M_n = \rho f_y bd^2 \left( 1 - 0.59 \frac{\rho f_y}{f_c'} \right)
\]  
(3.20b)
Failure by concrete crushing

If, for larger reinforcement ratios, the steel does not reach yield at failure, then the strain in the concrete becomes $\epsilon_u = 0.003$, as previously discussed. The steel stress $f_s$, not having reached the yield point, is proportional to the steel strain $\epsilon_s$; i.e., according to Hooke’s law,

$$f_s = \epsilon_s E_s$$

From the strain distribution of Fig. 3.6, the steel strain $\epsilon_s$ can be expressed in terms of the distance $c$ by evaluating similar triangles, after which it is seen that

$$f_s = \epsilon_u E_s \frac{d - c}{c}$$  \hspace{1cm} (3.21)

Then, from Eq. (3.16),

$$\alpha f'_c b c = A_s \epsilon_u E_s \frac{d - c}{c}$$  \hspace{1cm} (3.22)

Quadratic equation for $c$
Balanced reinforcement ratio $\rho_b$

\[ c = \frac{\varepsilon_u}{\varepsilon_u + \varepsilon_y} d \]  \hspace{1cm} (3.23)

Substituting that value of $c$ into Eq. (3.16), with $A_s f_s = \rho b d f_y$, one obtains for the balanced reinforcement ratio

\[ \rho_b = \frac{\alpha f'_c}{f_y} \frac{\varepsilon_u}{\varepsilon_u + \varepsilon_y} \]  \hspace{1cm} (3.24)
Example 3.3

Determine the nominal moment $M_n$ at which the beam of Examples 3.1 and 3.2 will fail.

**SOLUTION.** For this beam the reinforcement ratio $\rho = A_s/(bd) = 2.37/(10 \times 23) = 0.0103$. The balanced reinforcement ratio is found from Eq. (3.24) to be 0.0284. Since the amount of steel in the beam is less than that which would cause failure by crushing of the concrete, the beam will fail in tension by yielding of the steel. Its nominal moment, from Eq. (3.20b), is

$$M_n = 0.0103 \times 60,000 \times 10 \times 23^2 \left( 1 - 0.59 \frac{0.0103 \times 60,000}{4000} \right)$$

$$= 2,970,000 \text{ in-lb} = 248 \text{ ft-kips}$$

When the beam reaches $M_n$, the distance to its neutral axis, from Eq. (3.19b), is

$$c = \frac{0.0103 \times 60,000 \times 23}{0.72 \times 4000} = 4.94$$
It is informative to compare this result with those of Examples 3.1 and 3.2. In the previous calculations, it was found that at low loads, when the concrete had not yet cracked in tension, the neutral axis was located at a distance of 13.2 in. from the compression edge; at higher loads, when the tension concrete was cracked but stresses were still sufficiently small to be elastic, this distance was 7.6 in. Immediately before the beam fails, as has just been shown, this distance has further decreased to 4.9 in. For these same stages of loading, the stress in the steel increased from 2870 psi in the uncracked section, to 22,300 psi in the cracked elastic section, and to 60,000 psi at the nominal moment capacity. This migration of the neutral axis toward the compression edge and the increase in steel stress as load is increased is a graphic illustration of the differences between the various stages of behavior through which a reinforced concrete beam passes as its load is increased from zero to the value that causes it to fail. The examples also illustrate the fact that nominal moments cannot be determined accurately by elastic calculations.
Design of Tension-reinforced Rectangular Beams

- Demand < Capacity

- Hypothetical overload stage/demand with load factor
- Reduced capacity with strength reduction factor

- USD method of design - ACI 2008, BNBC 2015
- Limit states Design – Europe
- ULS, SLS limit states
Equivalent Rectangular Stress Distribution

**FIGURE 3.8**
 Actual and equivalent rectangular stress distributions at ultimate load.

\[ C = \alpha f'_c cb = \gamma f'_c ab \]

from which \( \gamma = \alpha \frac{c}{a} \)

C S Whitney
### TABLE 3.1
Concrete stress block parameters

<table>
<thead>
<tr>
<th></th>
<th>$f'_c$, psi</th>
</tr>
</thead>
<tbody>
<tr>
<td>≤4000</td>
<td>5000</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.72</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.425</td>
</tr>
<tr>
<td>$\beta_1 = 2\beta$</td>
<td>0.85</td>
</tr>
<tr>
<td>$\gamma = \alpha/\beta_1$</td>
<td>0.85</td>
</tr>
</tbody>
</table>

With $a = \beta_1 c$, this gives $\gamma = \alpha/\beta_1$. The second condition simply requires that in the equivalent rectangular stress block, the force $C$ be located at the same distance $\beta c$ from the top fiber as in the actual distribution. It follows that $\beta_1 = 2\beta$.

To supply the details, the upper two lines of Table 3.1 present the experimental evidence of Fig. 3.7 in tabular form. The lower two lines give the just-derived parameters $\beta_1$ and $\gamma$ for the rectangular stress block. It is seen that the stress intensity factor $\gamma$ is essentially independent of $f'_c$ and can be taken as 0.85 throughout. Hence, regardless of $f'_c$, the concrete compression force at failure in a rectangular beam of width $b$ is

$$C = 0.85f'_c ab$$  \hspace{1cm} (3.25)
Also, for the common concretes with $f'_c \leq 4000$ psi, the depth of the rectangular stress block is $a = 0.85c$, with $c$ being the distance to the neutral axis. For higher-strength concretes, this distance is $a = \beta_1 c$, with the $\beta_1$ values shown in Table 3.1. This is expressed in ACI Code 10.2.7.3 as follows: For $f'_c$ between 2500 and 4000 psi, $\beta_1$ shall be taken as 0.85; for $f'_c$ above 4000 psi, $\beta_1$ shall be reduced linearly at a rate of 0.05 for each 1000 psi of strength in excess of 4000 psi, but $\beta_1$ shall not be taken as less than 0.65. In mathematical terms, the relationship between $\beta_1$ and $f'_c$ can be expressed as

$$\beta_1 = 0.85 - 0.05 \frac{f'_c - 4000}{1000}$$

and

$$0.65 \leq \beta_1 \leq 0.85 \quad (3.26)$$
Balanced Strain condition

\[ c = \frac{\epsilon_u}{\epsilon_u + \epsilon_y} d \]  \hspace{1cm} (3.27)

which is seen to be identical to Eq. (3.23). Then from the equilibrium requirement that \( C = T \)

\[ \rho_b f_y b d = 0.85 f'_c ab = 0.85 \beta_1 f'_c bc \]

from which

\[ \rho_b = 0.85 \beta_1 \frac{f'_c}{f_y} \frac{\epsilon_u}{\epsilon_u + \epsilon_y} \]  \hspace{1cm} (3.28)

This is easily shown to be equivalent to Eq. (3.24).
Underreinforced beam

- Compression failure is abrupt
- Tensile failure gradual
- \( \rho \) should be less than \( \rho_b \)

Read points why?
\( \rho \) should be less than \( \rho_b \), why?

In actual practice, the upper limit on \( \rho \) should be below \( \rho_b \) for the following reasons: (1) for a beam with \( \rho \) exactly equal to \( \rho_b \), the compressive strain limit of the concrete would be reached, theoretically, at precisely the same moment that the steel reaches its yield stress, without significant yielding before failure; (2) material properties are never known precisely; (3) strain-hardening of the reinforcing steel, not accounted for in design, may lead to a brittle concrete compression failure even though \( \rho \) may be somewhat less than \( \rho_b \); (4) the actual steel area provided, considering standard reinforcing bar sizes, will always be equal to or larger than required, based on selected reinforcement ratio \( \rho \), tending toward overreinforcement; and (5) the extra ductility provided by beams with lower values of \( \rho \) increases the deflection capability substantially and thus provides warning prior to failure.
ACI provisions for underreinforced beam

- ACI establishes some safe limits
- Net tensile strain $\varepsilon_t$ at farthest from comp face
- Strength reduction factor $\varphi$
the reinforcement $d$. Substituting $d_t$ for $d$ and $\epsilon_t$ for $\epsilon_y$ in Eq. (3.27), the net tensile strain may be represented as

$$\epsilon_t = \epsilon_u \frac{d_t - c}{c} \quad (3.29)$$

Then based on Eq. (3.28), the reinforcement ratio to produce a selected value of net tensile strain is

$$\rho = 0.85\beta_1 \frac{f'_c}{f_y} \frac{d_t}{d} \frac{\epsilon_u}{\epsilon_u + \epsilon_t} \quad (3.30a)$$

or somewhat conservatively

$$\rho = 0.85\beta_1 \frac{f'_c}{f_y} \frac{\epsilon_u}{\epsilon_u + \epsilon_t} \quad (3.30b)$$

To ensure underreinforced behavior, ACI Code 10.3.5 establishes a minimum net tensile strain $\epsilon_t$ at the nominal member strength of 0.004 for members subjected to axial loads less than $0.10f'_cA_g$, where $A_g$ is the gross area of the cross section. By way of comparison $\epsilon_y$, the steel strain at the balanced condition, is 0.00207 for $f_y = 60,000$ psi and 0.00259 for $f_y = 75,000$ psi.

Using $\epsilon_t = 0.004$ in Eq. (3.30b) provides the maximum reinforcement ratio allowed by the ACI Code for beams

$$\rho_{\text{max}} = 0.85\beta_1 \frac{f'_c}{f_y} \frac{\epsilon_u}{\epsilon_u + 0.004} \quad (3.30c)$$
\[ \rho_{0.005} = 0.85\beta_1 \frac{f_c'}{f_y} \frac{\epsilon_u}{\epsilon_u + 0.005} \]  

(3.30d)

A comparison of Eqs. (3.30c) and (3.30d) shows that, for a given concrete cross section, using \( \epsilon_t = 0.004 \) will result in a higher reinforcement ratio, and thus a higher nominal flexural strength, than using \( \epsilon_t = 0.005 \). This higher strength, however, cannot be used to full advantage in design because the increase in flexural strength is canceled by the drop in \( \phi \) as \( \epsilon_t \) decreases from 0.005 to 0.004. As a result, the maximum practical reinforcement ratio for beams is attained at a net tensile strain of 0.005. Values of \( \epsilon_t \) below 0.005 are not recommended for the design of members with low axial loads.
FIGURE 3.9
Variation of strength reduction factor with net tensile strain in the steel.

\[ \phi = 0.75 + (\epsilon_t - 0.002) \times 50 \]
\[ \phi = 0.75 + 0.15 \left[ \frac{1}{(c/d_t)} - \frac{5}{3} \right] \]

Spiral

Other

\[ \phi = 0.65 + (\epsilon_t - 0.002) \times \frac{250}{3} \]
\[ \phi = 0.65 + 0.25 \left[ \frac{1}{(c/d_t)} - \frac{5}{3} \right] \]
FIGURE 3.10
Net tensile strain and $c/d_t$ ratios.

\[ \frac{c}{d_t} = \frac{0.003}{0.003 + 0.005} = 0.375 \]

(a) Tension-controlled member

\[ \frac{c}{d_t} = \frac{0.003}{0.003 + 0.004} = 0.429 \]

(b) Minimum net tensile strain for flexural member

\[ \frac{c}{d_t} = \frac{0.003}{0.003 + 0.002} = 0.600 \]

(c) Compression-controlled member
FIGURE 3.11
Singly reinforced rectangular beam.

that steel is yielding in tension, \( f_s = f_y \) at failure, and the nominal flexural strength (referring to Fig. 3.11) is given by

\[
M_n = A_s f_y \left( d - \frac{a}{2} \right)
\]  

(3.31)

where

\[
a = \frac{A_s f_y}{0.85 f'_c b}
\]  

(3.32)
It is convenient for everyday design to combine Eqs. (3.31) and (3.32) as follows. Noting that \( A_s = \rho b d \), Eq. (3.32) can be rewritten as
\[
a = \frac{\rho f_y d}{0.85 f'_c}
\]  
(3.33)

This is then substituted into Eq. (3.31) to obtain
\[
M_n = \rho f_y b d^2 \left( 1 - 0.59 \frac{\rho f_y}{f'_c} \right)
\]  
(3.34)

which is identical to Eq. (3.20b) derived in Section 3.3c. This basic equation can be simplified further as follows:
\[
M_n = R b d^2
\]  
(3.35)
in which
\[
R = \rho f_y \left( 1 - 0.59 \frac{\rho f_y}{f'_c} \right)
\]  
(3.36)

The flexural resistance factor \( R \) depends only on the reinforcement ratio and the
\[ \phi M_n = \phi A_s f_y \left( d - \frac{a}{2} \right) \] (3.37)

or, alternatively,

\[ \phi M_n = \phi \rho f_y b d^2 \left( 1 - 0.59 \frac{\rho f_y}{f'_c} \right) \] (3.38)

or

\[ \phi M_n = \phi R b d^2 \] (3.39)
EXAMPLE 3.4 Using the equivalent rectangular stress distribution, directly calculate the nominal strength of the beam previously analyzed in Example 3.3. Recall that \( b = 10 \) in., \( d = 23 \) in., \( A_s = 2.37 \) in\(^2\), \( f'_c = 4000 \) psi, \( f_y = 60,000 \) psi, and \( \beta_1 = 0.85 \).

SOLUTION. The distribution of stresses, internal forces, and strains is shown in Fig. 3.11. The maximum practical reinforcement ratio is calculated from Eq. (3.30d) as

\[
\rho_{0.005} = 0.85 \times 0.85 \times \frac{4000}{60,000} \times \frac{0.003}{0.003 + 0.005} = 0.0181
\]

and comparison with the actual reinforcement ratio of 0.0103 confirms that the member is underreinforced and will fail by yielding of the steel. Alternatively, recalling that \( c = 4.94 \) in.,

\[
\frac{c}{d_t} = \frac{c}{d} = \frac{4.94}{23} = 0.215
\]

which is less than 0.375, the value of \( c/d_t \) corresponding to \( \epsilon_t = 0.005 \), also confirming that the member is underreinforced. The depth of the equivalent stress block is found from the equilibrium condition that \( C = T \). Hence \( 0.85 f'_c ab = A_s f_y \), or \( a = 2.37 \times 60,000 \times (0.85 \times 4000 \times 10) = 4.18 \). The nominal moment is

\[
M_n = A_s f_y \left( d - \frac{a}{2} \right) = 2.37 \times 60,000 \times (23 - 2.09) = 2,970,000 \text{ in-lb} = 248 \text{ ft-kips}
\]
EXAMPLE 3.4 (continued)  Calculate the design moment capacity $\phi M_n$ for the beam analyzed earlier in Example 3.4.

SOLUTION. Comparing $\rho$ with $\rho_{0.005}$ or $c/d$, for the beam with the value of $c/d$, corresponding to $\varepsilon_f = 0.005$ demonstrates that $\varepsilon_f > 0.005$. Therefore, $\phi = 0.90$ and the design capacity is

$$\phi M_n = 0.9 \times 248 = 223 \text{ ft-kips}$$
Minimum Reinforcement Ratio

• If the flexural strength (of cracked section) is less than the moment that produced cracking of the previously uncracked section, the beam fails immediately upon formation of first flexural crack.

• To ensure against this type of failure, a minimum amount of reinforcement is provided
For a rectangular section having width $b$, total depth $h$, and effective depth $d$ (see Fig. 3.2b), the section modulus with respect to the tension fiber is $bh^2/6$. For typical cross sections, it is satisfactory to assume that $h/d = 1.1$ and that the internal lever arm at flexural failure is $0.95d$. If the modulus of rupture is taken as $f_r = 7.5 \sqrt{f'_c}$, as usual, then an analysis equating the cracking moment to the flexural strength results in

$$A_{s, \text{min}} = \frac{1.6 \sqrt{f'_c}}{f_y} bd \quad (3.40a)$$
This development can be generalized to apply to beams having a T cross section (see Section 3.8 and Fig. 3.16). The corresponding equations depend on the proportions of the cross section and on whether the beam is bent with the flange (slab) in tension or in compression. For T beams of typical proportions that are bent with the flange in compression, analysis will confirm that the minimum steel area should be

\[ A_{s,\text{min}} = \frac{2.7 \sqrt{f_c'}}{f_y} b_w d \quad (3.40b) \]

where \( b_w \) is the width of the web, or stem, projecting below the slab. For T beams that are bent with the flange in tension, from a similar analysis, the minimum steel area is

\[ A_{s,\text{min}} = \frac{6.2 \sqrt{f_c'}}{f_y} b_w d \quad (3.40c) \]
The ACI Code requirements for minimum steel area are based on the results just discussed, but there are some differences. According to ACI Code 10.5, at any section where tensile reinforcement is required by analysis, with some exceptions as noted below, the area $A_s$ provided must not be less than

$$A_{s, \text{min}} = \frac{3 \sqrt{f'_c}}{f_y} b_w d \geq \frac{200 b_w d}{f_y}$$  \hspace{1cm} (3.41)$$

According to ACI Code 10.5, the requirements of Eq. (3.41) need not be imposed if, at every section, the area of tensile reinforcement provided is at least one-third greater than that required by analysis. This provides sufficient reinforcement for large members such as grade beams, where the usual equations would require excessive amounts of steel.

For structural slabs and footings of uniform thickness, the minimum area of tensile reinforcement in the direction of the span is that required for shrinkage and temperature steel (see Section 13.3 and Table 13.2), and the above minimums need not be imposed. The maximum spacing of such steel is the smaller of 3 times the total slab thickness or 18 in.
EXAMPLE 3.5  Flexural strength of a given member. A rectangular beam has width 12 in. and effective depth 17.5 in. It is reinforced with four No. 9 (No. 29) bars in one row. If $f_y = 60,000$ psi and $f'_c = 4000$ psi, what is the nominal flexural strength, and what is the maximum moment that can be utilized in design, according to the ACI Code?

SOLUTION. From Table A.2 of Appendix A, the area of four No. 9 (No. 29) bars is 4.00 in$^2$. Assuming that the beam is underreinforced and using Eq. (3.32),

$$a = \frac{4.00 \times 60}{0.85 \times 4 \times 12} = 5.88 \text{ in.}$$

The depth of the neutral axis is $c = a/\beta_1 = 5.88/0.85 = 6.92$, giving

$$\frac{c}{d_t} = \frac{6.92}{17.5} = 0.395$$

which is between 0.429 and 0.375, the values corresponding, respectively, to $\epsilon_t = 0.004$ and $\epsilon_t = 0.005$, as shown in Fig. 3.10. Thus, the beam is, as assumed, underreinforced, and from Eq. (3.31)

$$M_n = 4.00 \times 60 \left(17.5 - \frac{5.88}{2}\right) = 3490 \text{ in-kips}$$

The fact that the beam is unreinforced could also have been established by calculating $\rho = 4.00/(12 \times 17.5) = 0.190$, which just exceeds $\rho_{0.005}$, which is calculated using Eq. (3.30d).

$$\rho_{0.005} = 0.85 \times 0.85 \left(\frac{4}{60}\right) \left(\frac{0.003}{0.003 + 0.005}\right) = 0.0181$$

Because the net tensile strain $\epsilon_t$ is between 0.004 and 0.005, $\phi$ must be calculated: $\epsilon_t = \epsilon_u (d - c)/c = 0.003 \times 17.5 - 6.92/6.92 = 0.00458$. Using linear interpolation from Fig. 3.9, $\phi = 0.87$, and the design strength is taken as

$$\phi M_n = 0.87 \times 3490 = 3040 \text{ in-kips}$$
The ACI Code limits on the reinforcement ratio

\[ \rho_{\text{max}} = 0.0206 \]

\[ \rho_{\text{min}} = \frac{3 \sqrt{4000}}{60,000} \geq \frac{200}{60,000} = 0.0033 \]

are satisfied for this beam.
EXAMPLE 3.6  Concrete dimensions and steel area to resist a given moment. Find the concrete cross section and the steel area required for a simply supported rectangular beam with a span of 15 ft that is to carry a computed dead load of 1.27 kips/ft and a service live load of 2.15 kips/ft, as shown in Fig. 3.12. Material strengths are $f'_c = 4000$ psi and $f_y = 60,000$ psi.

**SOLUTION.**  Load factors are first applied to the given service loads to obtain the factored load for which the beam is to be designed, and the corresponding moment:

$$ w_u = 1.2 \times 1.27 + 1.6 \times 2.15 = 4.96 \text{ kips/ft} $$

$$ M_u = \frac{1}{8} \times 4.96 \times 15^2 \times 12 = 1670 \text{ in-kips} $$

The concrete dimensions will depend on the designer’s choice of reinforcement ratio. To minimize the concrete section, it is desirable to select the maximum permissible reinforcement ratio. To maintain $\phi = 0.9$, the maximum reinforcement ratio corresponding to a net tensile strain of 0.005 will be selected (see Fig. 3.9). Then, from Eq. (3.30d)

$$ \rho_{0.005} = 0.85 \beta_1 \frac{f'_c}{f_y} \frac{\varepsilon_u}{\varepsilon_u + 0.005} = 0.85 \times 0.85 \left( \frac{4}{60} \right) \left( \frac{0.003}{0.003 + 0.005} \right) = 0.0181 $$

Using Eq. (3.30c) gives $\rho_{\text{max}} = 0.0206$, but would require a lower strength reduction factor. Setting the required flexural strength equal to the design strength from Eq. (3.38), and substituting the selected values for $\rho$ and material strengths,
\[ M_u = \phi M_n \]
\[ 1670 = 0.90 \times 0.0181 \times 60bd^2 \left(1 - 0.59 \frac{0.0181 \times 60}{4}\right) \]

from which
\[ bd^2 = 2040 \text{ in}^3 \]

A beam with width \( b = 10 \text{ in.} \) and \( d = 14.3 \text{ in.} \) will satisfy this requirement. The required steel area is found by applying the chosen reinforcement ratio to the required concrete dimensions:
\[ A_s = 0.0181 \times 10 \times 14.3 = 2.59 \text{ in}^2 \]

Two No. 10 (No. 32) bars provide 2.54 in\(^2\), which is very close to the required area.

Assuming 2.5 in. concrete cover from the centroid of the bars, the required total depth is \( h = 16.8 \text{ in.} \). In actual practice, however, the concrete dimensions \( b \) and \( h \) are always rounded up to the nearest inch, and often to the nearest multiple of 2 in. (see Section 3.5). The actual \( d \) is then found by subtracting the required concrete cover dimension from \( h \). For the present example, \( b = 10 \text{ in.} \) and \( h = 18 \text{ in.} \) will be selected, resulting in effective depth \( d = 15.5 \text{ in.} \).

Improved economy then may be possible, refining the steel area based on the actual, larger, effective depth. One can obtain the revised steel requirement directly by solving Eq. (3.38) for \( \rho \), with \( \phi M_n = M_u \). A quicker solution can be obtained by iteration. First a reasonable value of \( a \) is assumed, and \( A_s \) is found from Eq. (3.37). From Eq. (3.32) a revised estimate of \( a \) is obtained, and \( A_s \) is revised. This method converges very rapidly. For example, assume \( a = 5 \text{ in.} \). Then
\[ A_s = \frac{1670}{0.90 \times 60(15.5 - 2.5)} = 2.38 \text{ in}^2 \]

Checking the assumed \( a \) gives

\[ a = \frac{2.38 \times 60}{0.85 \times 4 \times 10} = 4.20 \text{ in.} \]

This is close enough to the assumed value that no further calculation is required. The required steel area of 2.38 in\(^2\) could be provided using three No. 8 (No. 25) bars, but for simplicity of construction, two No. 10 (No. 32) bars will be used as before.

A somewhat larger beam cross section using less steel may be more economical, and will tend to reduce deflections. As an alternative solution, the beam will be redesigned with a lower reinforcement ratio of \( \rho = 0.60 \rho_{\text{max}} = 0.60 \times 0.0206 = 0.0124 \). Setting the required strength equal to the design strength [Eq. (3.38)] as before,

\[ 1670 = 0.90 \times 0.0124 \times 60bd^2 \left(1 - 0.59 \frac{0.0124 \times 60}{4}\right) \]

and

\[ bd^2 = 2800 \text{ in}^3 \]

A beam with \( b = 10 \) in. and \( d = 16.7 \) in. will meet the requirement, for which

\[ A_s = 0.0124 \times 10 \times 16.7 = 2.07 \text{ in}^2 \]

Two No. 9 (No. 29) bars are almost sufficient, providing an area of 2.00 in\(^2\). If the total concrete height is rounded up to 20 in., a 17.5 in. effective depth results, reducing the required steel area to 1.96 in\(^2\). Two No. 9 (No. 29) bars remain the best choice.
• Infinite number of solution is possible

• Economic $0.5\rho_{0.005}$ to $0.75\rho_{0.005}$
**EXAMPLE 3.7**  **Determination of steel area.** Using the same concrete dimensions as were used for the second solution of Example 3.6 \((b = 10 \text{ in.}, d = 17.5 \text{ in.}, \text{ and } h = 20 \text{ in.})\) and the same material strengths, find the steel area required to resist a moment \(M_u\) of 1300 in-kips.

**SOLUTION.** Assume \(a = 4.0 \text{ in.}\). Then

\[
A_s = \frac{1300}{0.90 \times 60(17.5 - 2.0)} = 1.55 \text{ in}^2
\]

Checking the assumed \(a\) gives

\[
a = \frac{1.55 \times 60}{0.85 \times 4 \times 10} = 2.74 \text{ in.}
\]

Next assume \(a = 2.6 \text{ in.}\) and recalculate \(A_s\):

\[
A_s = \frac{1300}{0.90 \times 60(17.5 - 1.3)} = 1.49 \text{ in}^2
\]

No further iteration is required. Use \(A_s = 1.49 \text{ in}^2\). Two No. 8 (No. 25) bars, \(A_s = 1.58 \text{ in}^2\), will be used. A check of the reinforcement ratio shows \(\rho < \rho_{0.005}\) and \(\phi = 0.9\).
EXAMPLE 3.8  Determination of steel area and variable strength reduction factor.  Architectural considerations limit the height of a 20 ft long simple span beam to 16 in. and the width to 12 in. The following loads and material properties are given: \( w_d = 0.79 \text{ kips/ft}, \ w_l = 1.65 \text{ kips/ft}, \ f'_c = 5000 \text{ psi}, \) and \( f_y = 60,000 \text{ psi}. \) Determine the reinforcement for the beam.

SOLUTION.  Calculating the factored loads gives

\[
w_u = 1.2 \times 0.79 + 1.6 \times 1.65 = 3.59 \text{ kips/ft}
\]

\[
M_u = 3.59 \times \frac{20^2}{8} = 179 \text{ ft-kips} = 2150 \text{ in-kips}
\]

Assume \( a = 4.0 \text{ in.} \) and \( \phi = 0.90. \) The structural depth is \( (16 - 2.5) \text{ in.} = 13.5 \text{ in.} \) Calculating \( A_s \) gives

\[
A_s = \frac{M_u/\phi}{f_y(d - a/2)} = \frac{2150/0.90}{60(13.5 - 2.0)} = 3.46 \text{ in}^2
\]

Try two No. 10 (No. 32) and one No. 9 (No. 29) bar, \( A_s = 3.54 \text{ in}^2. \)

Check \( a = 3.54 \times 60/(0.85 \times 5 \times 12) = 4.16 \text{ in.} \) from Eq. (3.32). This is more than assumed; therefore, continue to check the moment capacity.

\[
M_n = 3.54 \times 60(13.5 - 4.16/2) = 2426 \text{ in-kips}
\]
Using a $\phi$ of 0.90 gives $\phi M_n = 2183$ in-kips, which is adequate; however, the net tensile strain must be checked to validate the selection of $\phi = 0.9$. In this case $c = a/\beta_1 = 4.16/0.80 : 5.20$ in. The $c/d$ ratio is $0.385 > 0.375$, so $\epsilon_t > 0.005$ is not satisfied. The corresponding net tensile strain is

$$
\epsilon_t = 0.003 \frac{13.5 - 5.2}{5.2} = 0.00479
$$

A value of $\epsilon_t = 0.00479$ is allowed by the ACI Code, but only if the strength reduction factor adjusted. A linear interpolation from Fig. 3.9 gives $\phi = 0.88$ and $M_u = \phi M_n = 2140$ in-kip which is less than the required capacity. Try increasing the reinforcement to three No. 10 (No. 3: bars, $A_s = 3.81$ in$^2$. Repeating the calculations,

$$
a = \frac{3.81 \times 60}{0.85 \times 5 \times 12} = 4.48 \text{ in.}
$$

$$
c = \frac{4.48}{0.80} = 5.60 \text{ in.}
$$

$$
M_n = 3.81 \times 60 \left(13.5 - \frac{4.48}{2}\right) = 2574 \text{ in-kips}
$$

$$
\epsilon_t = \frac{0.003(13.5 - 5.60)}{5.60} = 0.00423
$$

$$
\phi = 0.483 + 83.3 \times 0.00423 = 0.835
$$

$$
M_u = \phi M_n = 0.835 \times 2574 = 2150 \text{ in-kips}
$$

which meets the design requirements.
In actuality, the first solution deviates less than 1 percent from the desired value and would likely be acceptable. The remaining portion of the example demonstrates the design implications of requiring a variable strength reduction factor when the net tensile strain falls between 0.005 and 0.004. In this example, the reinforcement increased nearly 8 percent, yet the design moment capacity $\phi M_n$ only increased 0.5 percent due to the decreasing strength reduction factor. For this reason, designs with $\rho < \rho_{0.005}$ are desirable.
Overreinforced beam

According to the ACI Code, all beams are to be designed for yielding of the tension steel with $\varepsilon_t$ not less than 0.004 and thus $\rho \leq \rho_{\text{max}}$. Occasionally, however, such as when analyzing the capacity of existing construction, it may be necessary to calculate the flexural strength of an overreinforced compression-controlled member, for which $f_s$ is less than $f_y$ at flexural failure.

In this case, the steel strain, in Fig. 3.11b, will be less than the yield strain, but can be expressed in terms of the concrete strain $\varepsilon_u$ and the still-unknown distance $c$ to the neutral axis:

$$\varepsilon_s = \varepsilon_u \frac{d - c}{c}$$  \hspace{1cm} (3.42)
From the equilibrium requirement that \( C = T \), one can write

\[
0.85\beta_1 f'_c bc = \rho \epsilon_s E_s bd
\]

Substituting the steel strain from Eq. (3.42) in the last equation, and defining \( k_u = c/d \), one obtains a quadratic equation in \( k_u \) as follows:

\[
k_u^2 + m\rho k_u - m\rho = 0
\]

Here, \( \rho = A_s/bd \) as usual, and \( m \) is a material parameter given by

\[
m = \frac{E_s \epsilon_u}{0.85\beta_1 f'_c}
\]  \hspace{1cm} (3.43)

Solving the quadratic equation for \( k_u \),

\[
k_u = \sqrt{m\rho + \left(\frac{m\rho}{2}\right)^2} - \frac{m\rho}{2}
\]  \hspace{1cm} (3.44)

The neutral axis depth for the overreinforced beam can then easily be found from \( c = k_u d \), after which the stress-block depth \( a = \beta_1 c \). With steel strain \( \epsilon_s \) then computed from Eq. (3.42), and with \( f_s = E_s \epsilon_s \), the nominal flexural strength is

\[
M_n = A_s f_s \left( d - \frac{a}{2} \right)
\]  \hspace{1cm} (3.45)

The strength reduction factor \( \phi \) will equal 0.65 for beams in this range.
EXAMPLE 3.9  **Flexural strength of a given member.** Find the nominal flexural strength and design strength of the beam in Example 3.5, which has $b = 12$ in. and $d = 17.5$ in. and is reinforced with four No. 9 (No. 29) bars. Make use of the design aids of Appendix A. Material strengths are $f'_c = 4000$ psi and $f_y = 60,000$ psi.

**Solution.** From Table A.2, four No. 9 (No. 29) bars provide $A_s = 4.00$ in$^2$, and with $b = 12$ in. and $d = 17.5$ in., the reinforcement ratio is $\rho = 4.00/(12 \times 17.5) = 0.0190$. According to Table A.4, this is below $\rho_{\text{max}} = 0.0206$ and above $\rho_{\text{min}} = 0.0033$. Then from Table A.5b, with $f'_c = 4000$ psi, $f_y = 60,000$ psi, and $\rho = 0.019$, the value $R = 949$ psi is found. The nominal and design strengths are (with $\phi = 0.87$ from Example 3.5), respectively,

\[
M_n = Rbd^2 = 949 \times 12 \times \frac{17.5^2}{1000} = 3490 \text{ in-kips}
\]

\[
\phi M_n = 0.87 \times 3490 = 3040 \text{ in-kips}
\]
as before.
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\[ \rho = 0.85 \beta_1 \frac{f_y'}{f_y} \frac{0.003}{0.003 + \epsilon_t} \]

\[ \frac{c}{d_t} = 0.375, \frac{a}{d_t} = 0.375 \beta_1 \]

\[ \frac{c}{d_t} = 0.429, \frac{a}{d_t} = 0.429 \beta_1 \]
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### TABLE A.5b
Flexural resistance factor: \( R = \rho f_y \left( 1 - 0.588 \frac{\rho f_y}{f_c} \right) \) psi

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EXAMPLE 3.10  Concrete dimensions and steel area to resist a given moment. Find the cross section of concrete and the area of steel required for the beam in Example 3.6, making use of the design aids of Appendix A. \( M_u = 1670 \text{ in-kips} \), \( f'_c = 4000 \text{ psi} \), and \( f_y = 60,000 \text{ psi} \). Use a reinforcement ratio of \( 0.60 \rho_{\text{max}} \).

**Solution.** From Table A.4, the maximum reinforcement ratio is \( \rho_{\text{max}} = 0.0206 \). For economy, a value of \( \rho = 0.60 \rho_{\text{max}} = 0.0124 \) will be used. For that value, by interpolation from Table A.5a, the required value of \( R \) is 663. Then

\[
bd^2 = \frac{M_u}{\phi R} = \frac{1670 \times 1000}{0.90 \times 663} = 2800 \text{ in}^3
\]

Concrete dimensions \( b = 10 \text{ in.} \) and \( d = 16.7 \text{ in.} \) will satisfy this, but the depth will be rounded to 17.5 in. to provide a total beam depth of 20.0 in. It follows that

\[
R = \frac{M_u}{\phi bd^2} = \frac{1670 \times 1000}{0.90 \times 10 \times 17.5^2} = 606 \text{ psi}
\]

and from Table A.5a, by interpolation, \( \rho = 0.0112 \). This leads to a steel requirement of \( A_r = 0.0112 \times 10 \times 17.5 = 1.96 \text{ in}^2 \) as before.
**EXAMPLE 3.11**  

**Determination of steel area.** Find the steel area required for the beam in Example 3.7, with concrete dimensions $b = 10$ in. and $d = 17.5$ in. known to be adequate to carry the factored load moment of 1300 in-lb. Material strengths are $f'_c = 4000$ psi and $f_y = 60,000$ psi.

**Solution.** Note that in cases in which the concrete dimensions are known to be adequate and only the reinforcement must be found, the iterative method used earlier is not required. The necessary flexural resistance factor is

$$R = \frac{M_u}{\phi bd^2} = \frac{1300 \times 1000}{0.90 \times 10 \times 17.5^2} = 472 \text{ psi}$$

According to Table A.5a, with the specified material strengths, this corresponds to a reinforcement ratio of $\rho = 0.0085$, giving a steel area of

$$A_s = 0.0085 \times 10 \times 17.5 = 1.49 \text{ in}^2$$

as before. Two No. 8 (No. 25) bars will be used.
Practical considerations in the design of Beams: Concrete Protection for reinforcement

• Protection for steel against fire and corrosion
• Concrete cover depends on member and exposure
• Surfaces not exposed to ground or weather
  – Not less than $\frac{3}{4}$ in for slab
  – Not less than 1.5 in for beams and columns
• Surfaces exposed to weather or in contact with ground
  – At least 2 in
• Cast against ground with no form work
  – Min 3 in cover
FIGURE 3.13
Requirements for concrete cover in beams and slabs.
• b and h are rounded to 1 or 2 inch
• Slab rounded to \(\frac{1}{4}\) or \(\frac{1}{2}\) inch (greater than 6 inch)

• Proportions- d 2-3 times of b
Selection of bar and spacing

- No 3 to No 11 for beams
- No 14 and No 18 for columns

- Mixing of sizes allowed with 2 bar sizes
Gap between bars

- Clear distance between bars not less than bar dia or 1 inch
- Two or more layers - min 1 inch
- Upper bar directly above
DOUBLY REINFORCED BEAM

• Beams with tension and compression reinforcement
• Cross section is limited
• Compression steel is used for other reasons—long term deflection, reversal of moment, hanger bar for stirrup
Tension and compression steel both at yields

**FIGURE 3.14**
Doubly reinforced rectangular beam.
\[ M_{n1} = A'_s f_y (d - d') \] (3.46a)

as shown in Fig. 3.14d. The second part, \( M_{n2} \), is the contribution of the remaining tension steel \( A_s - A'_s \) acting with the compression concrete:

\[ M_{n2} = (A_s - A'_s)f_y \left( d - \frac{a}{2} \right) \] (3.46b)

as shown in Fig. 3.14e, where the depth of the stress block is

\[ a = \frac{(A_s - A'_s)f_y}{0.85 f'_c b} \] (3.47a)

With the definitions \( \rho = A_s / bd \) and \( \rho' = A'_s / bd \), this can be written

\[ a = \frac{(\rho - \rho')f_y d}{0.85 f'_c} \] (3.47b)

The total nominal resisting moment is then

\[ M_n = M_{n1} + M_{n2} = A'_s f_y (d - d') + (A_s - A'_s)f_y \left( d - \frac{a}{2} \right) \] (3.48)
\[ \bar{\rho}_b = \rho_b + \rho' \]  

(3.49)

where \( \rho_b \) is the balanced reinforcement ratio for the corresponding singly reinforced beam and is calculated from Eq. (3.28). The ACI Code limits the net tensile strain, not the reinforcement ratio. To provide the same margin against brittle failure as for singly reinforced beams, the maximum reinforcement ratio should be limited to

\[ \bar{\rho}_{\text{max}} = \rho_{\text{max}} + \rho' \]  

(3.50a)

Because \( \rho_{\text{max}} \) establishes the location of the neutral axis, the limitation in Eq. (3.50a) will provide acceptable net tensile strains. A check of \( \epsilon_t \) is required to determine the strength reduction factor \( \phi \) and to verify net tensile strain requirements are satisfied. Substituting \( \rho_{0.005} \) for \( \rho_{\text{max}} \) in Eq. (3.50a) will give the maximum reinforcement ratio for \( \phi = 0.90 \).

\[ \bar{\rho}_{0.005} = \rho_{0.005} + \rho' \]  

(3.50b)
Compression steel below yield stress

The preceding equations, through which the fundamental analysis of doubly reinforced beams is developed clearly and concisely, are valid only if the compression steel has yielded when the beam reached its nominal capacity. In many cases, such as for wide, shallow beams, beams with more than the usual concrete cover over the compression bars, beams with high yield strength steel, or beams with relatively small amounts of tensile reinforcement, the compression bars will be below the yield stress at failure. It is necessary, therefore, to develop more generally applicable equations to account for the possibility that the compression reinforcement has not yielded when the doubly reinforced beam fails in flexure.

Whether or not the compression steel will have yielded at failure can be determined as follows. Referring to Fig. 3.14b, and taking as the limiting case $\epsilon'_s = \epsilon_y$, one obtains, from geometry,

$$\frac{c}{d'} = \frac{\epsilon_u}{\epsilon_u - \epsilon_y} \quad \text{or} \quad c = \frac{\epsilon_u}{\epsilon_u - \epsilon_y} d'$$

Summing forces in the horizontal direction (Fig. 3.14c) gives the minimum tensile reinforcement ratio $\bar{\rho}_{cy}$ that will ensure yielding of the compression steel at failure:

$$\bar{\rho}_{cy} = 0.85 \beta_1 \frac{f'_c}{f_y} \frac{d'}{d} \frac{\epsilon_u}{\epsilon_u - \epsilon_y} + \rho'$$  \hspace{1cm} (3.51)
If the tensile reinforcement ratio is less than this limiting value, the neutral axis is sufficiently high that the compression steel stress at failure is less than the yield stress. In this case, it can easily be shown on the basis of Fig. 3.14b and c that the balanced reinforcement ratio is

\[ \bar{\rho}_b = \rho_b + \rho' \frac{f'_s}{f_y} \]  \hspace{1cm} (3.52)

where

\[ f'_s = E_s \varepsilon'_s = E_s \left[ \varepsilon_u - \frac{d'}{d}(\varepsilon_u + \varepsilon_y) \right] \leq f_y \]  \hspace{1cm} (3.53a)

To determine \( \rho_{\text{max}} \), \( \varepsilon_t = 0.004 \) is substituted for \( \varepsilon_y \) in Eq. (3.53a), giving

\[ f'_s = E_s \left[ \varepsilon_u - \frac{d'}{d}(\varepsilon_u + 0.004) \right] \leq f_y \]  \hspace{1cm} (3.53b)

Likewise, for \( \varepsilon_t = 0.005 \),

\[ f'_s = E_s \left[ \varepsilon_u - \frac{d'}{d}(\varepsilon_u + 0.005) \right] \leq f_y \]  \hspace{1cm} (3.53c)
Hence, the maximum reinforcement ratio permitted by the ACI Code is

\[
\bar{\rho}_{\text{max}} = \rho_{\text{max}} + \rho' \frac{f'_s}{f_y}
\]  

(3.54a)

and the maximum reinforcement ratio for \( \phi = 0.90 \) is

\[
\bar{\rho}_{0.005} = \rho_{0.005} + \rho' \frac{f'_s}{f_v}
\]  

(3.54b)
If the tensile reinforcement ratio is less than $\bar{\rho}_b$, as given by Eq. (3.52), and less than $\bar{\rho}_{cy}$, as given by Eq. (3.51), then the tensile steel is at the yield stress at failure but the compression steel is not, and new equations must be developed for compression steel stress and flexural strength. The compression steel stress can be expressed in terms of the still-unknown neutral axis depth as

$$f'_s = \varepsilon_u E_s \frac{c - d'}{c} \quad (3.55)$$

Consideration of horizontal force equilibrium (Fig. 3.14c with compression steel stress equal to $f'_s$) then gives

$$A_s f_y = 0.85\beta_1 f'_c bc + A'_s \varepsilon_u E_s \frac{c - d'}{c} \quad (3.56)$$

This is a quadratic equation in $c$, the only unknown, and is easily solved for $c$. The nominal flexural strength is found using the value of $f'_s$ from Eq. (3.55), and $a = \beta_1 c$ in the expression

$$M_n = 0.85f'_c ab \left( d - \frac{a}{2} \right) + A'_s f'_s (d - d') \quad (3.57)$$
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<th>$\epsilon_t = 0.005$</th>
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<td>Minimum $d$ for $d' = 2.5$ in., in.</td>
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Example 3.12

Flexural strength of a given member. A rectangular beam, shown in Fig. 3.15, has a width of 12 in. and an effective depth to the centroid of the tension reinforcement of 24 in. The tension reinforcement consists of six No. 10 (No. 32) bars in two rows. Compression reinforcement consisting of two No. 8 (No. 25) bars is placed 2.5 in. from the compression face of the beam. If $f_y = 60,000$ psi and $f_c' = 5000$ psi, what is the design moment capacity of the beam?

Solution. The steel areas and ratios are

$$A_s = 7.62 \text{ in}^2 \quad \rho = \frac{7.62}{12 \times 24} = 0.0265$$

$$A'_s = 1.58 \text{ in}^2 \quad \rho' = \frac{1.58}{12 \times 24} = 0.0055$$

Check the beam first as a singly reinforced beam to see if the compression bars can be disregarded,

$$\rho_{\text{max}} = 0.0243 \quad \text{from Table A.4 or Eq. (3.30c)}$$

The actual $\rho = 0.0265$ is larger than $\rho_{\text{max}}$, so the beam must be analyzed as doubly reinforced. From Eq. (3.51), with $\beta_1 = 0.80$,

$$\bar{\rho}_c = 0.85 \times 0.80 \times \frac{5}{60} \times \frac{2.5}{24} \times \frac{0.003}{0.003 - 0.00207} + 0.0055 = 0.0245$$
The tensile reinforcement ratio is greater than this, so the compression bars will yield when the beam fails. The maximum reinforcement ratio thus can be found from Eq. (3.50),

\[ \bar{\rho}_{\text{max}} = 0.0243 + 0.0055 = 0.0298 \]

The actual tensile reinforcement ratio is below the maximum value, as required. Then, from Eq. (3.47a),

\[ a = \frac{(7.62 - 1.58)60}{0.85 \times 5 \times 12} = 7.11 \text{ in.} \]

\[ c = \frac{a}{\beta_1} = \frac{7.11}{0.80} = 8.9 \text{ in.} \]

\[ \varepsilon_t = 0.003 \left( \frac{24 - 8.9}{8.89} \right) = 0.0051 \]
and

\[ \phi = 0.90 \]

and from Eq. (3.48),

\[ M_n = 1.58 \times 60(24 - 2.5) + 6.04 \times 60 \left( 24 - \frac{7.11}{2} \right) = 9450 \text{ in-kips} \]

The design strength is

\[ \phi M_n = 0.90 \times 9450 = 8500 \text{ in-kips} \]
Doubly Reinforced Beam

1. Calculate $\rho$, $\rho'$, and $(\rho - \rho')$
   
   $P_{\text{max}}$, $P_{\text{min}}$

2. Calculate
   
   $\bar{\rho}_c = 0.85 \beta_1 \frac{f'_c}{f_y} \frac{d'}{d} \frac{\varepsilon_u}{\varepsilon_u - 3\varepsilon_y} + \rho'$

3. If $\rho \geq \bar{\rho}_c$
   
   Compression steel yields, $f'_s = f_y$
   
   If $\rho < \bar{\rho}_c$
   
4. If comp steel yields, then
   
   a. Check that $P_{\text{max}} \geq (\rho - \rho') \geq P_{\text{min}}$

   b. Calculate
      
      $a = \left( \frac{A_s - A_{s'}}{f_y} \right) \frac{0.85f'_c}{b}$

   c. Calculate
      
      $M_n = a \left[ (A_s - A_{s'}) f_y (d - \frac{a}{2}) + A_{s'} f_y (d - d') \right] \frac{0.85f'_c}{b}$
5. If comp. steel does not yield, then

\[ T = C \]

\[ a. \quad T = 0.85 f' + b + \beta_1 c + A's f's \]

\[ A's f'y = 0.85 \beta_1 b c f' + A's E_s E_u \frac{c - d'}{c} \]

\[ \Rightarrow \text{Solve this quadratic equation to find } c \]

\[ b. \quad \text{Find } f's = E_s E_u \frac{c - d'}{c} \]

\[ c. \quad \text{Check } P_{\text{max}} \geq (P - P_{\text{min}} + \frac{f'y}{f's}) \geq P_{\text{min}} \]

\[ d. \quad \text{Calculate } a = \frac{A's f'y - A's f's}{0.85 f' b} \quad \text{or } a = \beta_1 c \quad \text{(check)} \]

\[ e. \quad \text{Calculate } \phi M_n = \phi \left[ (A's f'y - A's f's) (d - a/2) + A's f's (d - d') \right] \]
Find $\varphi M_n$
Nilson
Ex 3.12

Design moment capacity = ?

\[ f'_c = 5000 \text{ psi} \]
\[ f_y = 60,000 \text{ psi} \]

\[ \rho = \frac{A_s}{bd} = 0.0265 \]
\[ \rho' = \frac{A'_s}{bd} = 0.0055 \]

\[ \rho_{max} = 0.85 \beta_i \frac{f'_c}{f_y} \frac{E_y}{E_y + 0.004} = 0.0243 \]
\[ \rho_{min} = \frac{3 \sqrt{f'_c}}{f_y} \geq 0.00354 \]

**Solution**

\[ A_s = 6 \times 1.27 = 7.62 \text{ in}^2 \]
\[ A'_s = 2 \times 0.79 = 1.58 \text{ in}^2 \]

\[ \rho_{max} = 0.85 \beta_i \frac{f'_c}{f_y} \frac{E_y}{E_y + 0.004} = 0.0243 \]
\[ \rho_{min} = 0.00354 \]
\[ P_{cg} = 0.85 \beta_1 \frac{f'}{f_y} \frac{d'}{d} \frac{\varepsilon_u}{\varepsilon_u - \varepsilon_y} + \rho' \]

\[ = 0.85 \times 0.8 \times \frac{5}{60} \times \frac{2.5}{24} \times \frac{0.003}{0.003 - 0.00207} + 0.0055 = 0.0245 \]

\[ \rho > P_{cg} \implies \text{Comp bars yield when beam fails.} \]

\[ P_{max} = 0.0243 \ \text{N} \]

\[ \rho_{min} = 0.00354 \]

\[ \rho - \rho' = 0.021 \]

\[ a = \frac{A_s - A_{s'}}{0.85f'} \]

\[ f_y = 7.11 \]

\[ c = \frac{a}{\beta_1} = 8.88 \]

\[ \varepsilon_t = 0.003 \times \frac{24 - 8.88}{8.88} = 0.0051 \implies \phi = 0.9 \]

\[ M_n = A_{s'}f_y(d - d') + (A_s - A_{s'})f_y(d - 9/2) \]

\[ = 1.58 \times 60 (24 - 2.5) + 6.04 \times 60 (24 - \frac{7.11}{2}) = 9447 \ \text{kin} \]

\[ \Phi M_n = 8503 \ \text{K-in} \]
Nadim
Ex 3.10
2.5

Find $\Phi_M$
$V_c = 5$ ksi; $+y = 60$ ksi.

$A_s' = 3 \times 0.79 = 2.37$ in$^2$; $\rho' = 0.007524$
$A_s = 6 \times 1.27 = 7.62$ in$^2$; $\rho = 0.0242$
$\rho - \rho' = 0.01667$

$P_{\text{max}} = 0.0243 \quad \Rightarrow \quad (\rho - \rho') \text{ is ok.}$

$P_{\text{min}} = 0.00354$

2. $\bar{\rho}_{\text{cy}} = 0.85 \times 0.8 + \frac{5}{22.5} \times \frac{2.5 \times 0.003}{0.03 - 0.00207} + 0.007524 = 0.02783$

$P < \bar{\rho}_{\text{cy}} \Rightarrow \text{Comp bar does not yield}$

(to be more correct)
3. \[ T = C = C_s + C_c \]

\[ \Rightarrow A_{sfy} = 0.85f'_y \beta_{c,b} + A_5 E_s E_u \frac{c-d'}{c} = 0.85f'_y A_s \]

\[ \Rightarrow 7.62 + 60 = 0.85 + 5.8 + C + 2.37 + 29 + 10^{3} \cdot 0.03 \frac{c-2.5}{c} \]

\[ = 85 + 5 + 2.37 \]

\[ \Rightarrow 457.2 = 47.6C + 206.19 \frac{c-2.5}{c} - 10.07 \]

\[ \Rightarrow 47.6C - 261.08C - 515.475 = 0 \]

\[ C = \frac{261.08 \pm \sqrt{261.08^2 + 4 \cdot 47.6 \cdot 515.475}}{2 \cdot 47.6} \]

\[ = 7.026 \text{ in} \]

\[ a = \beta_1c = 5.62 \quad \Rightarrow f_s' = E_s E_u \frac{c-d'}{c} = 56.04 \text{ kpsi} \]
\[ \rho - \rho' \frac{f_s'}{f_y} = 0.0243 - 0.607524 + \frac{5604}{60} \]

\[ \Delta = 0.01724 < \rho_{\text{max}} \]

5. Find \( M_n \)

\[
M_n = (A's' - 0.85f'c'A's)(d - d') + (A's - A's'c')\left(d - \frac{a}{2}\right)
\]

\[
= (2.37 + 56 - 0.85 + 5 + 2.37) (22.5 - 2.5)
\]

\[
+ \left(7.62 - 60 - 2.37 + 56.04 + 85.6 + 2.37\right)
\]

\[
= 122.27 + 20.0 + 334.5 + 19.69
\]

\[
= 575.46 \text{ k.in.}
\]

Comp. in concrete = 0.85f'c'ab = \( C_c = 334.5 \text{ k} \)

Comp. in steel = \( A's'c' - \) force in displaced concrete

\[ C_s = A's'(f_s' - 0.85f'c') = 122.7 \text{ k} \]

Tension in steel = \( A's - f'c' = 45.72 \text{ k} \)

\[ T = C \text{ on.} \]

6. \[
\frac{E_t}{E_{tu}} = \frac{E_y}{C} \]

\[ E_t = \frac{0.03}{67.026} (23 - 7.026) \]

\[ = 0.00682 > 0.05 \]

\[ \phi = 0.9 \]

\[ C = \frac{22.5 + 5}{2} \]

\[ = 81.36 \text{ k.in.} \]

7. \( M_n = 81.36 \text{ k.in.} \)

\[ d_t = d + 1.0 \text{ if } d = h - 3.5 \]

\[ d_t = d + 1.5 \text{ if } d = h - 4.0 \]
Figure 3.27 Example 3.10 analysis solution.
Design of
Doubly Reinforced Beam
Design problem from Nadim

**Ex 4.5**  A beam section is limited to a width $b = 10$ in. and total depth of $h = 22$ in. and has to resist a factored moment of $226.5$ k-ft. Calculate the required reinforcement.

Given $f_{c'} = 3$ ksi and $f_y = 50$ ksi.

**Sol**

\[
\rho_{0.005} = 0.85 \beta_1 \frac{f_{c'}}{f_y} \frac{\sigma_u}{\sigma_u + 0.005} = 0.01625
\]

\[
A_s = 0.01625 \cdot 10 + 18.5 = 8 \text{ in}^2
\]

\[
a = \frac{A_s f_y}{0.85 f_{c'}} = 5.88 \text{ in}
\]

\[
d = 22 - 3.5 \text{ two layer } = 18.5
\]

\[
\sigma_u = \frac{226.5 \times 12}{2718} = 2718 \text{ k-in}
\]

\[
M_u = A_s f_y (d - \frac{a}{2}) = 2333.8 \text{ k-in}
\]

\[
\phi M_u = 0.9 \times 2333.8 = 2100.4 \text{ k-in}
\]

Doubly Reinforced beam reqd.
\[ \phi_{Mn_1} = 2100.4 \]
\[ \phi_{Mn_2} = 2718 - 2100.4 = 617.6 \text{ k}\]

Assuming comp. steel yield

\[ A_{s2} = \frac{617.6}{0.9 \times 50 (18.5 - 2.5)} = \frac{617.6}{0.9 \times 50 (16)} = 0.86 \text{ in}^2 \]

Total tension steel = \( A_s = 3.0 + 0.86 = 3.86 \text{ in}^2 \)

Comp. steel \( A_{s'} = 0.86 \text{ in}^2 \)

\[ \rho' = 0.00465 \]
\[ \rho^* = 0.02086 \]

\[ \rho - \rho' = 0.01621 \]
\[ \bar{\rho}_{cy} = 0.85\beta, \frac{f_c'}{f_y} d' \frac{d}{\frac{\varepsilon_u}{\varepsilon_y}} + \rho' \]
\[ = 0.85 + 0.85 \frac{3}{50} \frac{2.5}{18.5} \frac{0.03}{0.03 - 0.001724} + 0.00465 \]
\[ = 0.01842 \]
\[ \rho > \bar{\rho}_{cy} \text{ comp steel yields.} \]
\[ a = \frac{A_5 - A_{5'}}{0.85f_c'b} f_y = \frac{3.86 - 0.86}{0.85 + 3 + 10} = 5.88 \text{ in.} \]
\[ c = \frac{9}{81} = 6.92 \text{ in.} \]
\[ \varepsilon_t = \varepsilon_u \frac{d_t - c}{c} = 0.003 \frac{19.5 - 6.92}{6.92} = 0.00545 \]
\[ \phi = 0.9 \text{ Jr.} \]

![Diagram](image)
Design

Simply supported span = 18'

Beam section ⇒ 10" x 20"

\( f_c' = 4000 \text{ psi} \quad f_y = 60,000 \text{ psi} \)

find reinforcement.

\[
\begin{align*}
W_u &= 1.2(\text{LL}) + 1.05(\text{DL}) + 2.47 = 5212 \text{ k"/"

M_u &= \frac{1}{8} W_u \cdot \text{L}^2 = \frac{1}{8} \times 5212 \times 18^2 = 211.09 \text{ k"} = 2533 \text{ k"}

d &= 20 - 4 = 16" \text{ (two layer)}

d' &= 2.5" \text{ (if needed)}

\text{First check if possible to design singly reinforced.}

\varepsilon_t = 0.005

P_{0.005} = 0.0181

A_s = \rho bd = 0.0181 \times 10 \times 16 = 2.89 \text{ in}^2

\frac{a}{d'} = 6"
\[ A_s = \rho bd = 0.0181 + 10 + 10 = 20.0181 \text{ in}^2 \]

\[ a = \frac{A_{sf}y}{85f_{c}b} = 5.1 \text{ in} \quad c = \frac{a}{\rho} = 6'' \]

\[ M_n = A_{sf}y(d - \frac{a}{2}) = 2.89 + 60 \left(16 - \frac{5.1}{2}\right) = 2332 \text{ k}'' \]

\[ \phi M_n = 0.9 + 2332 = 2099 \text{ k}'' < M_u \quad \text{Doubly Rein. reqd.} \]

Remaining moment \( \phi M_n = 2533 - 2099 = 434 \text{ k}'' \)

\[ M_n = 482.4 \text{ k}'' \]

Assuming comp. bars yield

\[ A_{s2} = \frac{434}{0.9 + 60 \times (16 - 2.5)} = 0.6 \text{ in}^2 \]

Total tension steel = 2.89 + 0.6 = 3.49 in

Comp. steel = 0.6 in

4 # 9
4 in.

2 # 6
0.88 in.
\[ P = \frac{4}{10 \times 16} = 0.025 \quad \rho' = \frac{0.88}{10 \times 16} = 0.0055 \]

\[ \rho_y = 0.85 \beta \frac{f'_c}{f_y} \frac{d'}{d} \frac{E_u}{E_u - E_y} + \rho' \]

\[ = 0.85 + 0.85 \times \frac{4}{60} = \frac{2.5}{16} \times \frac{0.03}{0.03 - 0.00207} + 0.0055 \]

\[ = 0.0298 \]

\[ P < \rho_y \Rightarrow \text{Comp steel does not yield} \]

\[ A_{sy} = 0.85 f'_c b \beta_{sc} + A_{sf's} - A_s + 0.85 f'_c \]

\[ 4 \times 60 = 0.85 \times 4 \times 10 + 0.85 c + A_s \left[ \frac{E_s E_u}{c - d'} - 0.85 f'_c \right] \]

\[ = 240 c = 28.9 c^2 - 2.992 c + (76.56 c - 191.4) \]

\[ = 28.9 c^2 - 166.43 c - 191.4 = 0 \]
\[ f_s' = E_s \varepsilon_u \left( \frac{c}{d'} \right) = 54.7 \text{ ksi} \]

\[ C_c = 0.85 f_c' b \beta_1 c = 194.8 \text{ k} \]

\[ C_s = A_s' \left[ f_s' - 0.85 f_c' \right] = 45.14 \text{ k} \]

\[ T = A_s f_y = 4 \times 60 = 240 \text{ k} \]

\[ M_n = C_c (d - \frac{a}{2}) + C_s (d - d') \]

\[ = 194.8 \left( 16 - \frac{5.73}{2} \right) + 45.14 \left( 16 - 2.5 \right) \]

\[ = 2558.5 + 609.44 = 3167.9 \text{ k} \]
\[ \varepsilon_t = \varepsilon_0 \frac{d_t - c}{c} = 0.003 \frac{17.5 - 6.74}{6.74} \]
\[ = 0.00479 \]

\[ \phi = 0.483 + 83.3 \varepsilon_t = 0.882 \]

\[ N_n = 2793.9^k > N_u = 2533^k \text{ cm} \]
T-beam
• RC beam and slab are monolithically cast
• Beam stirrups and bent bars extend into the slab
• A part of slab act along with beam top to take longitudinal compression
• Slab forms the beam flange
• Part of beam below slab is called web/stem
Effective flange width

The criteria for effective width given in ACI Code 8.12 are as follows:

1. For symmetric T beams, the effective width $b$ shall not exceed one-fourth the span length of the beam. The overhanging slab width on either side of the beam web shall not exceed 8 times the thickness of the slab or go beyond one-half the clear distance to the next beam.

2. For beams having a slab on one side only, the effective overhanging slab width shall not exceed one-twelfth the span length of the beam, 6 times the slab thickness, or one-half the clear distance to the next beam.

3. For isolated beams in which the flange is used only for the purpose of providing additional compressive area, the flange thickness shall not be less than one-half the width of the web, and the total flange width shall not be more than 4 times the web width.
1. Symmetrical T beams
   
   \[ b < 16h_f + b_w \]
   
   \[ b < \text{Span/4} \]
   
   \[ b < c/c \text{ beam spacing} \]

2. Beam having slab on one side
   
   \[ b < \text{span/12} + b_w \]
   
   \[ b < 6h_f + b_w \]
   
   \[ b < \text{Half the clear span} + b_w \]

3. Isolated T beam
   
   \[ h_f > b_w / 2 \]
   
   \[ b < 4b_w \]
FIGURE 3.17
Effective flange width of T beams.
Strength Analysis

Two possibilities
• Just like rectangular beam
• T-beam analysis required
If $a > h_f$ T-beam

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{\rho f_y d}{0.85 f'_c} \quad (3.58)$$
\[ A_{sf} = \frac{0.85f'_c (b - b_w)h_f}{f_y} \]  
(3.59)

\[ M_{n1} = A_{sf}f_y\left(d - \frac{h_f}{2}\right) \]  
(3.60)

The remaining steel area \( A_s - A_{sf} \):

\[ a = \frac{(A_s - A_{sf})f_y}{0.85f'_c b_w} \]  
(3.61)

\[ M_{n2} = (A_s - A_{sf})f_y\left(d - \frac{a}{2}\right) \]  
(3.62)
and the total nominal resisting moment is the sum of the parts:

\[ M_n = M_{n1} + M_{n2} = A_{sf} f_y \left( d - \frac{h_f}{2} \right) + (A_s - A_{sf}) f_y \left( d - \frac{a}{2} \right) \]  

(3.63)
EXAMPLE 3.14

Moment capacity of a given section. The isolated T beam shown in Fig. 3.21 is composed of a flange 28 in. wide and 6 in. deep cast monolithically with a web of 10 in. width that extends 24 in. below the bottom surface of the flange to produce a beam of 30 in. total depth. Tensile reinforcement consists of six No. 10 (No. 32) bars placed in two horizontal rows separated by 1 in. clear spacing. The centroid of the bar group is 26 in. from the top of the beam. The concrete has a strength of 3000 psi, and the yield strength of the steel is 60,000 psi. What is the design moment capacity of the beam?
**SOLUTION.** It is easily confirmed that the flange dimensions are satisfactory according to the ACI Code for an isolated beam. The entire flange can be considered effective. For six No. 10 (No. 32) bars, $A_s = 7.62$ in$^2$. First check the location of the neutral axis, on the assumption that rectangular beam equations may be applied. Using Eq. (3.32)

$$a = \frac{7.62 \times 60}{0.85 \times 3 \times 28} = 6.40 \text{ in.}$$

This exceeds the flange thickness, and so a T beam analysis is required. From Eq. (3.59) and Fig. 3.19b,

$$A_{sf} = 0.85 \times \frac{3}{60} (28 - 10) \times 6 = 4.59 \text{ in}^2$$

Then, from Eq. (3.60),

$$M_{n1} = 4.59 \times 60 (26 - 3) = 6330 \text{ in-kips}$$

Then, from Fig. 3.19c,

$$A_s - A_{sf} = 7.62 - 4.59 = 3.03 \text{ in}^2$$

and from Eqs. (3.58) and (3.59)

$$a = \frac{3.03 \times 60}{0.85 \times 3 \times 10} = 7.13 \text{ in.}$$

$$M_{n2} = 3.03 \times 60 (26 - 3.56) = 4080 \text{ in-kips}$$
The depth to the neutral axis is \( c = a_1/\beta_1 = 7.13/0.85 = 8.39 \) and \( d_i = 27.5 \) in. to the lowest bar. The \( c/d_i \) ratio is \( 8.39/27.5 = 0.305 < 0.375 \), so the \( \epsilon_t > 0.005 \) requirement is met and \( \phi = 0.90 \). When the ACI strength reduction factor is incorporated, the design strength is

\[
\phi M_n = 0.90(6330 + 4080) = 9370 \text{ in-kips}
\]
EXAMPLE 3.15

Determination of steel area for a given moment. A floor system, shown in Fig. 3.22, consists of a 3 in. concrete slab supported by continuous T beams with a 24 ft span, 47 in. on centers. Web dimensions, as determined by negative-moment requirements at the supports, are $b_w = 11$ in. and $d = 20$ in. What tensile steel area is required at midspan to resist a factored moment of 6400 in-kips if $f_y = 60,000$ psi and $f'_c = 3000$ psi?
SOLUTION. First determining the effective flange width,

\[ 16h_f + b_w = 16 \times 3 + 11 = 59 \text{ in.} \]

\[ \frac{\text{Span}}{4} = 24 \times \frac{12}{4} = 72 \text{ in.} \]

Centerline beam spacing = 47 in.

The centerline T beam spacing controls in this case, and \( b = 47 \text{ in.} \). The concrete dimensions \( b_w \) and \( d \) are known to be adequate in this case, since they have been selected for the larger negative support moment applied to the effective rectangular section \( b_w d \). The tensile steel at midspan is most conveniently found by trial. Assuming the stress-block depth \( a \) is equal to the flange thickness of \( h_f = 3 \text{ in.} \), one gets

\[ d - \frac{a}{2} = 20 - 1.50 = 18.50 \text{ in.} \]

Trial:

\[ A_s = \frac{M_u}{\phi f_y (d - a/2)} = \frac{6400}{0.90 \times 60 \times 18.50} = 6.41 \text{ in}^2 \]

Checking the assumed value for \( a \),

\[ a = \frac{A_s f_y}{0.85 f'_c b} = \frac{6.41 \times 60}{0.85 \times 3 \times 47} = 3.21 \text{ in.} \]

Since \( a \) is greater than \( h_f \), a T beam design is required and \( \phi = 0.90 \) is assumed.

\[ A_{sf} = \frac{0.85 f'_c (b - b_w) h_f}{f_y} = \frac{0.85 \times 3 \times 36 \times 3}{60} = 4.59 \text{ in}^2 \]
\[
\phi M_{n1} = \phi A_{s} f_{y} \left( d - \frac{h_{f}}{2} \right) = 0.90 \times 4.59 \times 60 \times 18.50 = 4590 \text{ in-kips}
\]

\[
\phi M_{n2} = M_{u} - \phi M_{n1} = 6400 - 4590 = 1810 \text{ in-kips}
\]

Assume \( a = 4.00 \text{ in.} \):

\[
A_{s} - A_{s} = \frac{\phi M_{n2}}{\phi f_{y} (d - a/2)} = \frac{1810}{0.90 \times 60 \times (20 - 4.0/2)} = 1.86 \text{ in}^{2}
\]

Check:

\[
a = \frac{(A_{s} - A_{s}) f_{y}}{0.85 f'_{c} b_{w}} = \frac{1.86 \times 60}{0.85 \times 3 \times 11} = 3.98 \text{ in.}
\]

This is satisfactorily close to the assumed value of 4 in. Then

\[
A_{s} = A_{s} + A_{s} - A_{s} = 4.59 + 1.86 = 6.45 \text{ in}^{2}
\]

Checking to ensure that the net tensile strain of 0.005 is met to allow \( \phi = 0.90 \),

\[
c = \frac{a}{\beta_{1}} = \frac{3.98}{0.85} = 4.68
\]

\[
\frac{c}{d_{f}} = \frac{4.68}{20} = 0.23 < 0.325
\]

indicating that the design is satisfactory.

The close agreement should be noted between the approximate tensile steel area of 6.41 in\(^{2}\) found by assuming the stress-block depth equal to the flange thickness and the more exact value of 6.45 in\(^{2}\) found by T beam analysis. The approximate solution would be satisfactory in most cases.